

Cubic edge-transitive bi-Cayley graphs over inner-abelian p -groups

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Abstract

A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a group of automorphisms acting semiregularly on its vertices with two orbits. A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian. In this paper, we complete the classification of connected cubic edge-transitive bi-Cayley graphs over inner-abelian p -groups for an odd prime p .

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1 Introduction

Throughout this paper, we denote by \mathbb{Z}_n the cyclic group of order n and by \mathbb{Z}_n^* the multiplicative group of \mathbb{Z}_n consisting of numbers coprime to n . All groups are assumed to be finite, and all graphs are assumed to be finite, connected, simple and undirected. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertex set $V(\Gamma)$, and edge set $E(\Gamma)$. Denote by $\text{Aut}(\Gamma)$ the full automorphism group of Γ . For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ . For a graph Γ , if $\text{Aut}(\Gamma)$ is transitive on $V(\Gamma)$ or $E(\Gamma)$, then Γ is said to be *vertex-transitive* or *edge-transitive*, respectively. An arc-transitive graph is also called a symmetric graph.

Let G be a permutation group on a set Ω and take $\alpha \in \Omega$. The stabilizer G_α of α in G is the subgroup of G fixing the point α . The group G is said to be *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a semiregular automorphism group with two orbits (Bi-Cayley graph is sometimes called *semi-Cayley graph*). Note that every bi-Cayley graph admits the following concrete realization. Given a group H , let \mathcal{R} , \mathcal{L} and S be subsets of H such that $\mathcal{R}^{-1} = \mathcal{R}$, $\mathcal{L}^{-1} = \mathcal{L}$ and $\mathcal{R} \cup \mathcal{L}$ does not contain the identity element of H . The *bi-Cayley graph* over H relative to the triple $(\mathcal{R}, \mathcal{L}, S)$, denoted by $\text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$, is the graph having vertex set the union $H_0 \cup H_1$ of two copies of H , and edges of the form $\{h_0, (xh)_0\}$, $\{h_1, (yh)_1\}$ and $\{h_0, (zh)_1\}$

with $x \in \mathcal{R}, y \in \mathcal{L}, z \in S$ and $h_0 \in H_0, h_1 \in H_1$ representing a given $h \in H$. Let $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$. For $g \in H$, define a permutation $R(g)$ on the vertices of Γ by the rule

$$h_i^{R(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h \in H.$$

Then $R(H) = \{R(g) \mid g \in H\}$ is a semiregular subgroup of $\text{Aut}(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $R(H)$ is normal in $\text{Aut}(\Gamma)$, the bi-Cayley graph $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ is called a *normal bi-Cayley graph* over H (see [15]). A bi-Cayley graph $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ is called *normal edge-transitive* if $N_{\text{Aut}(\Gamma)}(R(H))$ is transitive on the edge-set of Γ (see [15]).

There are many important graphs which can be constructed as bi-Cayley graphs. For example, the Petersen graph is a bi-Cayley graph over a cyclic group of order 5. Another interesting bi-Cayley graph is the Gray graph [3] which is a bi-Cayley graph over a metacyclic 3-group of order 27. One more example of bi-Cayley graph is the Hoffman-Singleton graph [8] which is a bi-Cayley graph over an elementary abelian group of order 25. We note that all of these graphs are bi-Cayley graphs over a p -group. Inspired by this, we are naturally led to investigate the bi-Cayley graphs over a p -group.

In [15], a characterization is given of cubic edge-transitive bi-Cayley graphs over a 2-group. A next natural step would be studying cubic edge-transitive bi-Cayley graphs over a p -group, where p is an odd prime. Due to Zhou et al.'s work in [14] about the classification of cubic vertex-transitive abelian bi-Cayley graphs, we may assume the p -group in question is non-abelian. As the beginning of this program, in [12] we prove that every cubic edge-transitive bi-Cayley graph over a p -group is normal whenever $p > 7$, and moreover, it is shown that a cubic edge-transitive bi-Cayley graph over a metacyclic p -group exists only when $p = 3$, and cubic edge-transitive bi-Cayley graphs over a metacyclic p -group are normal except the Gray graph. Recall that a non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian. We note that the Gray graph [3], the smallest cubic semisymmetric graph, is isomorphic to $\text{BiCay}(H, \emptyset, \emptyset, \{1, a, a^2b\})$, where H is the following inner-abelian metacyclic group of order 27

$$\langle a, b \mid a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle.$$

In [12], a complete classification is given of cubic edge-transitive bi-Cayley graphs over an inner-abelian metacyclic p -group.

In this paper, we shall complete the classification of cubic edge-transitive bi-Cayley graphs over any inner-abelian p -group. By [13] or [1, Lemma 65.2], for every odd prime p , an inner-abelian non-metacyclic p -group is isomorphic to the following group:

$$\mathcal{H}_{p,t,s} = \langle a, b, c \mid a^{p^t} = b^{p^s} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle (t \geq s \geq 1). \quad (1)$$

Now we define a family of cubic bi-Cayley graphs over $\mathcal{H}_{p,t,s}$. If $t = s$, then take $k = 0$, while if $t > s$, take $k \in \mathbb{Z}_{p^{t-s}}^*$ such that $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$. Let

$$\Sigma_{p,t,s,k} = \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, \{1, a, ba^k\}). \quad (2)$$

It will be shown in Lemma 3.2 that for any two distinct admissible integers k_1, k_2 , the graphs Σ_{p,t,s,k_1} and Σ_{p,t,s,k_2} are isomorphic. So the graph $\Sigma_{p,t,s,k}$ is independent of the choice of k , and we denote by $\Sigma_{p,t,s}$ the graph $\Sigma_{p,t,s,k}$.

Before stating our main result, we introduce some symmetry properties of graphs. An s -arc, $s \geq 1$, in a graph Γ is an ordered $(s+1)$ -tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$, and a 1-arc is usually called an *arc*. A graph Γ is said to be s -arc-transitive if $\text{Aut}(\Gamma)$ is transitive on the set of s -arcs in Γ . An s -arc-transitive graph is said to be s -transitive if it is not $(s+1)$ -arc-transitive. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A subgroup G of $\text{Aut}(\Gamma)$ is s -arc-regular if for any two s -arcs of Γ , there is a unique element $g \in G$ mapping one to the other, and Γ is said to be s -arc-regular if $\text{Aut}(\Gamma)$ is s -arc-regular. It is well known that, in the cubic case, an s -transitive graph is s -arc-regular.

Theorem 1.1 *Let Γ be a connected cubic edge-transitive bi-Cayley graph over $\mathcal{H}_{p,s,t}$. Then $\Gamma \cong \Sigma_{p,t,s}$. Furthermore, the following hold:*

- (1) $\Sigma_{3,2,1}$ is 3-arc-regular;
- (2) $\Sigma_{p,t,s}$ is 2-arc-regular if $t = s$;
- (3) $\Sigma_{3,t,s}$ is 2-arc-regular if $t = s + 1$, and $(t, s) \neq (2, 1)$;
- (4) $\Sigma_{p,t,s}$ is 1-arc-regular if $p^{t-s} > 3$.

We shall close this section by introducing some notation which will be used in this paper. For a finite group G , the full automorphism group, the center, the derived subgroup and the Frattini subgroup of G will be denoted by $\text{Aut}(G)$, $Z(G)$, G' and $\Phi(G)$, respectively. For $x, y \in G$, denote by $o(x)$ the order of x and by $[x, y]$ the commutator $x^{-1}y^{-1}xy$. For a subgroup H of G , denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G . For two groups M and N , $N \rtimes M$ denotes a semidirect product of N by M .

2 Some basic properties of the group $\mathcal{H}_{p,t,s}$

In this section, we will give some properties of the group $\mathcal{H}_{p,t,s}$ (given in Equation (1)).

Lemma 2.1 *Let $H = \mathcal{H}_{p,t,s}$. Then the following hold:*

- (1) For any $i \in \mathbb{Z}_p^t$, we have $a^i b = b a^i c^i$.
- (2) $H' = \langle c \rangle \cong \mathbb{Z}_p$.
- (3) For any $x, y \in H$, we have $(xy)^p = x^p y^p$.
- (4) For any $x, y \in H$, if $o(x) = o(a) = p^t$, $o(y) = o(b) = p^s$ and $H = \langle x, y \rangle$, then H has an automorphism taking (a, b) to (x, y) .

(5) Every maximal subgroup of H is one of the following groups:

$$\begin{aligned}\langle ab^j, b^p, c \rangle &= \langle ab^j \rangle \times \langle b^p \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{s-1}} \times \mathbb{Z}_p (j \in \mathbb{Z}_p), \\ \langle a^p, b, c \rangle &= \langle a^p \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^{t-1}} \times \mathbb{Z}_{p^s} \times \mathbb{Z}_p.\end{aligned}$$

Proof For (1), for any $i \in \mathbb{Z}_{p^t}$, since $[a, b] = c$ and $[c, a] = 1$, we have $b^{-1}ab = ac$ and $ac = ca$, and then $b^{-1}a^ib = (b^{-1}ab)^i = (ac)^i = a^ic^i$. It follows that $a^ib = ba^ic^i$, and so (1) holds.

From [1, Lemma 65.2], we have the items (2) and (3).

For (4), assume that $H = \langle x, y \rangle$, and $o(x) = o(a), o(y) = o(b)$. Let $z = [x, y]$. Then $z \neq 1$, and then by (2), we have $H' = \langle z \rangle = \langle c \rangle$. It follows that $z^p = 1$ and $[z, x] = [z, y] = 1$. Consequently, x and y have the same relations as do a and b . Therefore, H has an automorphism taking (a, b) to (x, y) .

For (5), let M be a maximal subgroup of H . As H is a 2-generator group, we have $H/\Phi(H) = \langle a\Phi(H) \rangle \times \langle b\Phi(H) \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Clearly, $M/\Phi(H)$ is a subgroup of $H/\Phi(H)$ of order p , so $M/\Phi(H) = \langle ab^j\Phi(H) \rangle$ or $\langle b\Phi(H) \rangle$ for some $j \in \mathbb{Z}_p$. Note that $\Phi(H) = \langle a^p, b^p, c \rangle$ is contained in the center of H . It follows that M is one of the following groups:

$$\begin{aligned}\langle ab^j, b^p, c \rangle &= \langle ab^j \rangle \times \langle b^p \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{s-1}} \times \mathbb{Z}_p (j \in \mathbb{Z}_p), \\ \langle a^p, b, c \rangle &= \langle a^p \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^{t-1}} \times \mathbb{Z}_{p^s} \times \mathbb{Z}_p.\end{aligned}$$

This proves (5). □

3 The isomorphisms of $\Sigma_{p,t,s,k}$

The goal of this section is to prove the graph $\Sigma_{p,t,s,k}$ is independent on the choice of k . By the definition, if $t = s$, then $k = 0$, and so for any given group $\mathcal{H}_{p,t,s}$, we only have one graph. So we only need to consider the case when $t > s$. We first restate an easily proved result about bi-Cayley graphs.

Proposition 3.1 [14, Lemma 3.1] *Let $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over a group H . Then the following hold:*

- (1) H is generated by $\mathcal{R} \cup \mathcal{L} \cup S$.
- (2) Up to graph isomorphism, S can be chosen to contain the identity of H .
- (3) For any automorphism α of H , $\text{BiCay}(H, \mathcal{R}, \mathcal{L}, S) \cong \text{BiCay}(H, \mathcal{R}^\alpha, \mathcal{L}^\alpha, S^\alpha)$.

Lemma 3.2 *Suppose that $t > s$ and $k_1, k_2 \in \mathbb{Z}_{p^{t-s}}^*$ are two distinct solutions of the equation $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$. Then $\Sigma_{p,t,s,k_1} \cong \Sigma_{p,t,s,k_2}$.*

Proof Recall that

$$\mathcal{H}_{p,t,s} = \langle a, b, c \mid a^{p^t} = b^{p^s} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

and

$$\Sigma_{p,t,s,k_i} = \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, T_i), \text{ where } T_i = \{1, a, ba^{k_i}\} \text{ with } i = 1, 2.$$

We first show that there exists an automorphism β of $\mathcal{H}_{p,t,s}$ which sends (a, b) to $(ba^{k_2}, a(ba^{k_2})^{-k_1})$. It is easy to see that $ba^{k_2}, a(ba^{k_2})^{-k_1}$ generate $\mathcal{H}_{p,t,s}$. By Lemma 2.1 (4), it suffices to show that $o(a) = o(ba^{k_2})$ and $o(b) = o(a(ba^{k_2})^{-k_1})$. Since $k_2 \in \mathbb{Z}_{p^{t-s}}^*$, from Lemma 2.1 (3) it follows that $o(a) = o(ba^{k_2})$. By Lemma 2.1 (3), we have $(a(ba^{k_2})^{-k_1})^{p^s} = a^{p^s} (b^{p^s} a^{k_2 p^s})^{-k_1} = (a^{p^s})^{1-k_1 k_2}$. Since $k_1, k_2 \in \mathbb{Z}_{p^{t-s}}^*$ satisfy $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$, it follows that $-k_1, -k_2$ are two elements of $\mathbb{Z}_{p^{t-s}}^*$ of order 3. Since $\mathbb{Z}_{p^{t-s}}^*$ is cyclic, we have $k_1 k_2 \equiv 1 \pmod{p^{t-s}}$. Consequently, $(a^{p^s})^{1-k_1 k_2} = 1$ and so $o(a(ba^{k_2})^{-k_1}) = o(b)$.

Now we know that $\mathcal{H}_{p,t,s}$ has an automorphism β taking (a, b) to $(ba^{k_2}, a(ba^{k_2})^{-k_1})$. Moreover,

$$T_{k_1}^\beta = \{1, a, ba^{k_1}\}^\beta = \{1, ba^{k_2}, a(ba^{k_2})^{-k_1} \cdot (ba^{k_2})^{k_1}\} = \{1, ba^{k_2}, a\} = T_{k_2}.$$

By Proposition 3.1 (3), we have

$$\Sigma_{p,t,s,k_1} = \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, T_1) \cong \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, T_2) = \Sigma_{p,t,s,k_2},$$

as required. \square

4 The automorphisms of $\Sigma_{p,t,s}$

The topic of this section is the automorphisms of $\Sigma_{p,t,s}$.

4.1 Preliminaries

In this subsection, we give some preliminary results. Let Γ be a connected graph with an edge-transitive group G of automorphisms and let N be a normal subgroup of G . The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits. Below we introduce two propositions, of which the first is a special case of [9, Theorem 9].

Proposition 4.1 *Let Γ be a cubic graph and let $G \leq \text{Aut}(\Gamma)$ be arc-transitive on Γ . Then G is an s -arc-regular subgroup of $\text{Aut}(\Gamma)$ for some integer s . If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, Γ_N is a cubic symmetric graph with G/N as an s -arc-regular subgroup of automorphisms.*

The next proposition is a special case of [10, Lemma 3.2].

Proposition 4.2 *Let Γ be a cubic graph and let $G \leq \text{Aut}(\Gamma)$ be transitive on $E(\Gamma)$ but intransitive on $V(\Gamma)$. Then Γ is a bipartite graph with two partition sets, say V_0 and V_1 . If $N \trianglelefteq G$ is intransitive on each of V_0 and V_1 , then N is semiregular on $V(\Gamma)$, Γ_N is a cubic graph with G/N as an edge- but not vertex-transitive group of automorphisms.*

The following result gives an upper bound of the order of the vertex-stabilizer of cubic edge-transitive graphs.

Proposition 4.3 [11, Proposition 8] *Let Γ be a connected cubic edge-transitive graph and let $G \leq \text{Aut}(\Gamma)$ be transitive on the edges of Γ . For any $v \in V(\Gamma)$, the stabilizer G_v has order $2^r \cdot 3$ with $r \geq 0$.*

The next three propositions are about cubic edge-transitive bi-Cayley graphs over a p -group.

Proposition 4.4 [12, Lemma 4.1] *Let Γ be a connected cubic edge-transitive graph of order $2p^n$ with p an odd prime and $n \geq 2$. Let $G \leq \text{Aut}(\Gamma)$ be transitive on the edges of Γ . Then any minimal normal subgroup of G is an elementary abelian p -group.*

Proposition 4.5 [12, Lemma 4.2] *Let $p \geq 5$ be a prime and let Γ be a connected cubic edge-transitive graph of order $2p^n$ with $n \geq 1$. Let $A = \text{Aut}(\Gamma)$ and let H be a Sylow p -subgroup of A . Then Γ is a bi-Cayley graph over H , and moreover, if $p \geq 11$, then Γ is a normal bi-Cayley graph over H .*

Proposition 4.6 [12, Lemma 4.3] *Let Γ be a connected cubic edge-transitive graph of order $2p^n$ with $p = 5$ or 7 and $n \geq 2$. Let $Q = O_p(A)$ be the maximal normal p -subgroup of $A = \text{Aut}(\Gamma)$. Then $|Q| = p^n$ or p^{n-1} .*

4.2 Normality of cubic edge-transitive bi-Cayley graphs over $\mathcal{H}_{p,t,s}$

The following lemma determines the normality of cubic edge-transitive bi-Cayley graphs over $\mathcal{H}_{p,t,s}$.

Lemma 4.7 *Let Γ be a connected cubic edge-transitive bi-Cayley graph over $\mathcal{H}_{p,t,s}$. If $p = 3$, then Γ is normal edge-transitive. If $p > 3$, then Γ is normal.*

Proof Let $A = \text{Aut}(\Gamma)$ and let P be a Sylow p -subgroup of A such that $R(H) \leq P$. Let $H = \mathcal{H}_{p,t,s}$, and let $|H| = p^n$ with $n = t + s + 1$. If $p = 3$, then by Proposition 4.3, we have $|A| = 3^{n+1} \cdot 2^r$ with $r \geq 0$. This implies that $|P| = 3|R(H)|$, and so $|P_{1_0}| = |P_{1_1}| = 3$. Thus, P is transitive on the edges of Γ . Clearly, $R(H) \trianglelefteq P$. This implies that Γ is normal edge-transitive.

Suppose now $p > 3$. Then $R(H)$ is a Sylow p -subgroup of A . Suppose to the contrary that $R(H)$ is not normal in A . By Proposition 4.5, we have $p = 5$ or 7 . Let N be the maximal normal p -subgroup of A . Then $N \leq R(H)$, and by Proposition 4.6, we have

$|R(H) : N| = p$. By Propositions 4.1 and 4.2, the quotient graph Γ_N is a cubic graph of order $2p$ with A/N as an edge-transitive automorphism group. By [5, 6], if $p = 5$, then Γ_N is the Petersen graph, and if $p = 7$, then Γ_N is the Heawood graph. Since A/N is transitive on the edges of Γ_N and $R(H)/N$ is non-normal in A/N , it follows that

$$\begin{aligned} A_5 &\lesssim A/N \lesssim S_5, & \text{if } p = 5; \\ \text{PSL}(2, 7) &\lesssim A/N \lesssim \text{PGL}(2, 7), & \text{if } p = 7. \end{aligned}$$

Let B/N be the socle of A/N . Then B/N is also edge-transitive on Γ_N , and so B is also edge-transitive on Γ . Let $C = C_B(N)$. Then $C/(C \cap N) \cong CN/N \trianglelefteq B/N$. Since B/N is non-abelian simple, one has $CN/N = 1$ or B/N .

Suppose first that $CN/N = 1$. Then $C \leq N$, and so $C = C \cap N = C_N(N) = Z(N)$. Since $R(H)$ is inner-abelian, we have N is abelian, and so $C = Z(N) = N$. Recall that $|R(H) : N| = p$. Then N is a maximal subgroup of $R(H)$. By Lemma 2.1 (5), we have $N \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{s-1}} \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^{t-1}} \times \mathbb{Z}_{p^s} \times \mathbb{Z}_p$. Let $\mathcal{U}_1(N) = \{x^p \mid x \in N\}$ and $M = (R(H))'\mathcal{U}_1(N)$. Then $\mathcal{U}_1(N) \cong \mathbb{Z}_{p^{t-1}} \times \mathbb{Z}_{p^{s-2}}$ or $\mathbb{Z}_{p^{t-2}} \times \mathbb{Z}_{p^{s-1}}$. Moreover, M is characteristic in N and $N/M \cong \mathbb{Z}_p \times \mathbb{Z}_p$. It implies that each element g of B induces an automorphism of N/M , denote by $\sigma(g)$. Consider the map $\varphi : B \rightarrow \text{Aut}(N/M)$ with $\varphi(g) = \sigma(g)$ for any $g \in B$. It is easy to check that φ is a homomorphism. Letting $\text{Ker } \varphi$ be the kernel of φ , we have $\text{Ker } \varphi = C = N$. It follows that $B/N \lesssim \text{Aut}(N/M) \cong \text{GL}(2, p)$. This forces that either $A_5 \leq \text{GL}(2, 5)$ with $p = 5$, or $\text{PSL}(2, 7) \leq \text{GL}(2, 7)$ with $p = 7$. However, each of these can not happen by Magma [2], a contradiction.

Suppose now that $CN/N = B/N$. Since $C \cap N = Z(N)$, we have $1 < C \cap N \leq Z(C)$. Clearly, $Z(C)/(C \cap N) \trianglelefteq C/(C \cap N) \cong CN/N$. Since $CN/N = B/N$ is non-abelian simple, $Z(C)/C \cap N$ must be trivial. Thus $C \cap N = Z(C)$, and hence $B/N = CN/N \cong C/C \cap N = C/Z(C)$. If $C = C'$, then $Z(C)$ is a subgroup of the Schur multiplier of B/N . However, the Schur multiplier of A_5 or $\text{PSL}(2, 7)$ is \mathbb{Z}_2 , a contradiction. Thus, $C \neq C'$. Since $C/Z(C)$ is non-abelian simple, one has $C/Z(C) = (C/Z(C))' = C'Z(C)/Z(C) \cong C'/(C' \cap Z(C))$, and then we have $C = C'Z(C)$. It follows that $C'' = C'$. Clearly, $C' \cap Z(C) \leq Z(C')$, and $Z(C')/(C' \cap Z(C)) \trianglelefteq C'/(C' \cap Z(C))$. Since $C'/(C' \cap Z(C)) \cong C/Z(C)$ and since $C/Z(C)$ is non-abelian simple, it follows that $Z(C')/(C' \cap Z(C))$ is trivial, and so $Z(C') = C' \cap Z(C)$. As $C/(C \cap N) \cong CN/N$ is non-abelian, we have $C/(C \cap N) = (C/(C \cap N))' = (C/Z(C))' \cong C'/(C' \cap Z(C)) = C'/Z(C')$. Since $C' = C''$, $Z(C')$ is a subgroup of the Schur multiplier of CN/N . However, the Schur multiplier of A_5 or $\text{PSL}(2, 7)$ is \mathbb{Z}_2 , forcing that $Z(C') \cong \mathbb{Z}_2$. This is impossible because $Z(C') = C' \cap Z(C) \leq C \cap N$ is a p -subgroup. Thus $R(H) \trianglelefteq A$, as required. \square

4.3 Automorphisms of $\Sigma_{p,t,s}$

We first collect several results about the automorphisms of the bi-Cayley graph $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$. Recall that for each $g \in H$, $R(g)$ is a permutation on $V(\Gamma)$ defined by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2, h, g \in H, \quad (3)$$

and $R(H) = \{R(g) \mid g \in H\} \leq \text{Aut}(\Gamma)$. For an automorphism α of H and $x, y, g \in H$, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as following:

$$\begin{aligned} \delta_{\alpha,x,y} : & \quad h_0 \mapsto (xh^\alpha)_1, \quad h_1 \mapsto (yh^\alpha)_0, \quad \forall h \in H, \\ \sigma_{\alpha,g} : & \quad h_0 \mapsto (h^\alpha)_0, \quad h_1 \mapsto (gh^\alpha)_1, \quad \forall h \in H. \end{aligned} \quad (4)$$

Set

$$\begin{aligned} \text{I} &= \{\delta_{\alpha,x,y} \mid \alpha \in \text{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = x^{-1}\mathcal{L}x, \mathcal{L}^\alpha = y^{-1}\mathcal{R}y, S^\alpha = y^{-1}S^{-1}x\}, \\ \text{F} &= \{\sigma_{\alpha,g} \mid \alpha \in \text{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = \mathcal{R}, \mathcal{L}^\alpha = g^{-1}\mathcal{L}g, S^\alpha = g^{-1}S\}. \end{aligned} \quad (5)$$

Proposition 4.8 [15, Theorem 3.4] *Let $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over the group H . Then $N_{\text{Aut}(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$ and $N_{\text{Aut}(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\alpha,x,y} \rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:*

- (1) $\langle R(H), \delta_{\alpha,x,y} \rangle$ acts transitively on $V(\Gamma)$;
- (2) if α has order 2 and $x = y = 1$, then Γ is isomorphic to the Cayley graph $\text{Cay}(\bar{H}, \mathcal{R} \cup \alpha S)$, where $\bar{H} = H \rtimes \langle \alpha \rangle$.

Lemma 4.9 *The graph $\Sigma_{p,t,s}$ is symmetric.*

Proof Recall that

$$\mathcal{H}_{p,t,s} = \langle a, b, c \mid a^{p^t} = b^{p^s} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

and

$$\Sigma_{p,t,s} = \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, \{1, a, ba^k\}),$$

where if $t = s$, then $k = 0$, and if $t > s$, then $k \in \mathbb{Z}_{p^{t-s}}^*$ satisfies $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$.

We first prove the following two claims.

Claim 1 $\mathcal{H}_{p,t,s}$ has an automorphism α mapping a, b to $a^{-1}ba^k, a^{-1}(a^{-1}ba^k)^{-k}$, respectively.

By definition, if $t = s$ then $k = 0$, and by Lemma 2.1 (4), we can obtain Claim 1. Let $t > s$. Then $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$. Let $x = a^{-1}ba^k$ and $y = a^{-1}(a^{-1}ba^k)^{-k}$. Note that $(yx^k)^{-1} = a$ and $(yx^k)^{-1}x(yx^k)^k = b$. This implies that $\langle x, y \rangle = \langle a, b \rangle = \mathcal{H}_{p,t,s}$.

By Lemma 2.1 (1), we have $x = a^{-1}ba^k = ba^{k-1}c^{-1}$. Since $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$, we have $(k-1, p) = 1$. By Lemma 2.1 (3), we have $o(x) = o(a) = p^t$. Since $p^{t-s} \mid k^2 - k + 1$, again by Lemma 2.1 (3),

$$y^{p^s} = (a^{-1}(a^{-1}ba^k)^{-k})^{p^s} = a^{-p^s}(a^{-p^s}b^{p^s}a^{kp^s})^{-k} = (a^{-p^s})^{k^2-k+1} = 1,$$

and so $o(y) = o(b) = p^s$. By Lemma 2.1 (4), $\mathcal{H}_{p,t,s}$ has an automorphism taking (a, b) to (x, y) , as claimed.

Claim 2. $\mathcal{H}_{p,t,s}$ has an automorphism β mapping a, b to $a^{-1}, a^{-k}b^{-1}a^k$, respectively.

Let $u = a^{-1}$ and $v = a^{-k}b^{-1}a^k$. Clearly, $\langle u, v \rangle = \langle a, b \rangle = \mathcal{H}_{p,t,s}$ and $o(u) = p^t$. Note that

$$v^{p^s} = (a^{-k}b^{-1}a^k)^{p^s} = a^{-kp^s}b^{-p^s}a^{kp^s} = 1.$$

So $o(v) = o(b) = p^s$. By Lemma 2.1 (4), $\mathcal{H}_{p,t,s}$ has an automorphism taking (a, b) to (u, v) , as claimed.

Now we are ready to finish the proof of our lemma. Set $T = \{1, a, ba^k\}$. By Claim 1, there exists $\alpha \in \text{Aut}(\mathcal{H}_{p,t,s})$ such that $a^\alpha = a^{-1}ba^k$ and $b^\alpha = a^{-1}(a^{-1}ba^k)^{-k}$. Then

$$a^{-1}T = a^{-1}\{1, a, ba^k\} = \{a^{-1}, 1, a^{-1}ba^k\},$$

$$T^\alpha = \{1, a, ba^k\}^\alpha = \{1, a^{-1}ba^k, a^{-1}(a^{-1}ba^k)^{-k} \cdot (a^{-1}ba^k)^k\} = \{1, a^{-1}ba^k, a^{-1}\}.$$

Thus $T^\alpha = a^{-1}T$. By Proposition 4.8, $\sigma_{\alpha,a}$ is an automorphism of $\Sigma_{p,t,s}$ fixing 1_0 and cyclically permutating the three neighbors of 1_0 . Set $B = R(\mathcal{H}_{p,t,s}) \rtimes \langle \sigma_{\alpha,a} \rangle$. Then B acts transitively on the edges of $\Sigma_{p,t,s}$.

By Claim 2, there exists $\beta \in \text{Aut}(\mathcal{H}_{p,t,s})$ such that $a^\beta = a^{-1}$ and $b^\beta = a^{-k}b^{-1}a^k$. Then

$$T^\beta = \{1, a, ba^k\}^\beta = \{1, a^{-1}, a^{-k}b^{-1}a^k \cdot a^{-k}\} = \{1, a^{-1}, a^{-k}b^{-1}\} = T^{-1}.$$

By Proposition 4.8, $\delta_{\beta,1,1}$ is an automorphism of $\Sigma_{p,t,s}$ swapping 1_0 and 1_1 . Thus, $\Sigma_{p,t,s}$ is vertex-transitive, and so $\Sigma_{p,t,s}$ is symmetric. \square

Theorem 4.10 *One of the following holds.*

- (1) $\Sigma_{3,2,1}$ is 3-arc-regular;
- (2) $\Sigma_{p,t,s}$ is 2-arc-regular if $t = s$;
- (3) $\Sigma_{3,t,s}$ is 2-arc-regular if $t = s + 1$, and $(t, s) \neq (2, 1)$;
- (4) $\Sigma_{p,t,s}$ is 1-arc-regular if $p^{t-s} > 3$.

Proof By Magma [2], we can obtain (1). If $(p, t, s) = (3, 1, 1)$ then by Magma [2], we can show that $\Sigma_{3,1,1}$ is 2-arc-regular. In what follows, we assume that $(p, t, s) \neq (3, 2, 1), (3, 1, 1)$.

Set $\Gamma = \Sigma_{p,t,s}$ and $H = \mathcal{H}_{p,t,s}$. We shall first prove that Γ is a normal bi-Cayley graph over H . By Lemma 4.7, we may assume that $p = 3$. Since $(p, t, s) \neq (3, 1, 1), (3, 2, 1)$, one has $|H| = 3^{t+s+1} \geq 3^5$. Let $n = t + s + 1$.

Let $A = \text{Aut}(\Gamma)$ and let P be a Sylow 3-subgroup of A such that $R(H) \leq P$. Then $R(H) \trianglelefteq P$. By Lemma 4.9, Γ is symmetric. We first prove the following claim.

Claim 1 $P \trianglelefteq A$.

Let $M \trianglelefteq A$ be maximal subject to that M is intransitive on both H_0 and H_1 . By Proposition 4.1, M is semiregular on $V(\Gamma)$ and the quotient graph Γ_M of Γ relative to M is a cubic graph with A/M as an arc-transitive group of automorphisms. Assume that $|M| = 3^\ell$. Then $|V(\Gamma_M)| = 2 \cdot 3^{n-\ell}$. If $n - \ell \leq 3$, then by [5], Γ_M is isomorphic to F006A,

F018A or F054A, and then by Magma [2], $\text{Aut}(\Gamma_M)$ has a normal Sylow 3-subgroup. It follows that $P/M \trianglelefteq A/M$, and so $P \trianglelefteq A$, as claimed.

Now suppose that $n - \ell > 3$. Take a minimal normal subgroup N/M of A/M . By Proposition 4.4, N/M is an elementary abelian 3-group. By the maximality of M , N is transitive on at least one of H_0 and H_1 , and so $3^n \mid |N|$. If $3^{n+1} \mid |N|$, then $P = N \trianglelefteq A$, as claimed. Now assume that $|N| = 3^n$. We have N is transitive on both H_0 and H_1 . Then N is semiregular on both H_0 and H_1 , and then Γ_M would be a cubic bi-Cayley graph on N/M . Since Γ_M is connected, by Proposition 3.1, N/M is generated by two elements, and so $N/M \cong \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. This implies that $|V(\Gamma_M)| = 6$ or 18 , contrary to the assumption that $|V(\Gamma_M)| = 2 \cdot 3^{n-\ell} > 18$, completing the proof of our claim.

By Claim 1, we have $P \trianglelefteq A$. Since $|P : R(H)| = 3$, one has $\Phi(P) \leq R(H)$. As H is non-abelian, one has $\Phi(P) < R(H)$ for otherwise, we would have P is cyclic and so H is cyclic which is impossible. Then $\Phi(P)$ is intransitive on both H_0 and H_1 , the two orbits of $R(H)$ on $V(\Gamma)$. Since $\Phi(P)$ is characteristic in P , $P \trianglelefteq A$ gives that $\Phi(P) \trianglelefteq A$. By Propositions 4.1, the quotient graph $\Gamma_{\Phi(P)}$ of Γ relative to $\Phi(P)$ is a cubic graph with $A/\Phi(P)$ as an arc-transitive group of automorphisms. Furthermore, $P/\Phi(P)$ is transitive on the edges of $\Gamma_{\Phi(P)}$. Since $P/\Phi(P)$ is abelian, it is easy to see that $\Gamma_{\Phi(P)} \cong K_{3,3}$, and so $P/\Phi(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Let Φ_2 be the Frattini subgroup of $\Phi(P)$. Then $\Phi_2 \trianglelefteq A$ because Φ_2 is characteristic in $\Phi(P)$ and $\Phi(P) \trianglelefteq A$. Let Φ_3 be the Frattini subgroup of Φ_2 . Similarly, we have $\Phi_3 \trianglelefteq A$. Now we prove the following claim.

Claim 2 $\Phi(P)/\Phi_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Phi_2/\Phi_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since $P/\Phi(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $|P : R(H)| = 3$, we have $|R(H) : \Phi(P)| = 3$. Then $\Phi(P)$ is a maximal subgroup of $R(H)$. And then by Lemma 2.1 (5), we have $\Phi(P)$ is isomorphic to one of the following four groups:

$$\begin{aligned} M_1 &= \langle a \rangle \times \langle b^3 \rangle \times \langle c \rangle, & M_2 &= \langle a^3 \rangle \times \langle b \rangle \times \langle c \rangle, \\ M_3 &= \langle ab \rangle \times \langle b^3 \rangle \times \langle c \rangle, & M_4 &= \langle ab^{-1} \rangle \times \langle b^3 \rangle \times \langle c \rangle. \end{aligned}$$

It follows that $\Phi(P)/\Phi_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then Φ_2 is isomorphic to one of the following four groups:

$$\begin{aligned} Q_1 &= \langle a^3 \rangle \times \langle b^9 \rangle \cong \mathbb{Z}_{3^{t-1}} \times \mathbb{Z}_{3^{s-2}}, & Q_2 &= \langle a^9 \rangle \times \langle b^3 \rangle \cong \mathbb{Z}_{3^{t-2}} \times \mathbb{Z}_{3^{s-1}}, \\ Q_3 &= \langle a^3 b^3 \rangle \times \langle b^9 \rangle \cong \mathbb{Z}_{3^{t-1}} \times \mathbb{Z}_{3^{s-2}}, & Q_4 &= \langle a^3 b^{-3} \rangle \times \langle b^9 \rangle \cong \mathbb{Z}_{3^{t-1}} \times \mathbb{Z}_{3^{s-2}}. \end{aligned}$$

It implies that $\Phi_2/\Phi_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, as claimed.

Clearly, $\Phi_3 \leq \Phi(P) < R(H)$, so Φ_3 is intransitive on both H_0 and H_1 . Consider the quotient graph Γ_{Φ_3} of Γ relative to Φ_3 . By Propositions 4.1, Γ_{Φ_3} is a cubic graph with A/Φ_3 as an arc-transitive group of automorphisms. Furthermore, Γ_{Φ_3} is a bi-Cayley graph over the group $R(H)/\Phi_3$ of order $2 \cdot 3^6$. Then by [4], $\Gamma_{\Phi_3} \cong \text{C1458.1, C1458.2, C1458.3, C1458.4, C1458.5, C1458.6, C1458.7, C1458.8, C1458.9, C1458.10 or C1458.11}$. By Magma [2], if $\Gamma_{\Phi_3} \cong \text{C1458.1, C1458.3, C1458.4, C1458.8, C1458.9, C1458.10 or C1458.11}$, then $\text{Aut}(\Gamma_{\Phi_3})$ does not have an abelian or inner-abelian semiregular subgroup of order 729,

a contradiction. If $\Gamma_{\Phi_3} \cong \text{C1458.2, C1458.5, C1458.6 or C1458.7}$, then by Magma [2], all semiregular subgroups of $\text{Aut}(\Gamma_{\Phi_3})$ of order 729 are normal, and so $R(H)/\Phi_3 \trianglelefteq \text{Aut}(\Gamma_{\Phi_3})$. It follows that $R(H)/\Phi_3 \trianglelefteq A/\Phi_3$, and so $R(H) \trianglelefteq A$.

By now we have shown that $\Sigma_{p,t,s}$ is normal. By [7, Theorem 1.1], $\Sigma_{p,t,s}$ is at most 2-arc-transitive. Recall that Lemma 4.9 already proved that $\Sigma_{p,t,s}$ is at least 1-arc-transitive.

Let $t = s$. In this case, we have $k = 0$ and

$$\Sigma_{p,t,t} = \text{BiCay}(\mathcal{H}_{p,t,t}, \emptyset, \emptyset, \{1, a, b\}).$$

It is easy to see that $\mathcal{H}_{p,t,t}$ has an automorphism γ swapping a and b . Then $\sigma_{\gamma,1} \in \text{Aut}(\Sigma_{p,t,t})_{1_0 1_1}$ and $\sigma_{\gamma,1}$ swaps a_1 and b_1 . Thus, $\Sigma_{p,t,t}$ is 2-arc-regular.

Let $t = s + 1$ and $p = 3$. In this case, we have $k^2 - k + 1 \equiv 0 \pmod{3}$ and so $k = 2$ since $k \in \mathbb{Z}_3^*$. Then

$$\Sigma_{p,s+1,s} = \text{BiCay}(\mathcal{H}_{p,s+1,s}, \emptyset, \emptyset, \{1, a, ba^2\}).$$

By Lemma 2.1 (1), we see that $(ba^2)^2 = b^2 a^4 c^2$, and so $a(ba^2)^{-2} = a^{-3} b^{-2} c^{-2}$, which has the same order as b . Noticing that $o(a) = o(ba^2)$, by Lemma 2.1 (4), $\mathcal{H}_{p,s+1,s}$ has an automorphism γ taking (a, b) to $(ba^2, a(ba^2)^{-2})$. Furthermore, γ swaps a and ba^2 . Then $\sigma_{\gamma,1} \in \text{Aut}(\Sigma_{p,s+1,s})_{1_0 1_1}$ and $\sigma_{\gamma,1}$ swaps a_1 and $(ba^2)_1$. Thus, $\Sigma_{p,s+1,s}$ is 2-arc-regular.

Let $p^{t-s} > 3$. In this case, $\Sigma_{p,t,s} = \text{BiCay}(\mathcal{H}_{p,t,s}, \emptyset, \emptyset, \{1, a, ba^k\})$, where $k \in \mathbb{Z}_{p^{t-s}}^*$ satisfies $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$. If $\Sigma_{p,t,s}$ is 2-arc-regular, then by [7, Theorem 1.1], $\mathcal{H}_{p,t,s}$ has an automorphism γ swapping a and ba^k , and then $b^\gamma = (ba^k)^\gamma (a^{-k})^\gamma = a(ba^k)^{-k}$. It follows that $1 = (a(ba^k)^{-k})^{p^s} = (a^{p^s})^{1-k^2}$, and so $1 - k^2 \equiv 0 \pmod{p^{t-s}}$. Combining this with the equation $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$, we have $k \equiv 2 \pmod{p^{t-s}}$, forcing $p^{t-s} = 3$, a contradiction. Thus, $\Sigma_{p,t,s}$ is 1-arc-regular. \square

5 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1 To complete the proof, by Theorem 4.10, it suffices to prove that every cubic edge-transitive bi-Cayley graph over $\mathcal{H}_{p,t,s}$ is isomorphic to $\Sigma_{p,t,s}$.

Let $H = \mathcal{H}_{p,t,s}$, and let $\Gamma = \text{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected cubic edge-transitive bi-Cayley graph over $\mathcal{H}_{p,t,s}$. Set $A = \text{Aut}(\Gamma)$. By Lemma 4.7, we have Γ is normal edge-transitive. It follows that the two orbits H_0, H_1 of $R(H)$ on $V(\Gamma)$ do not contain edges of Γ , and so $\mathcal{R} = \mathcal{L} = \emptyset$. By Proposition 3.1, we may assume that $S = \{1, x, y\}$ for $x, y \in H$. Since Γ is connected, by Proposition 3.1, we have $H = \langle S \rangle = \langle x, y \rangle$.

Since Γ is normal edge-transitive, by Proposition 4.8, there exists $\sigma_{\alpha,h} \in A_{1_0}$, where $\alpha \in \text{Aut}(H)$ and $h \in H$, such that $\sigma_{\alpha,h}$ cyclically permutes the three elements in $\Gamma(1_0) = \{1_1, x_1, y_1\}$. Without loss of generality, assume that $(\sigma_{\alpha,h})|_{\Gamma(1_0)} = (1_1 \ x_1 \ y_1)$. Then $x_1 = (1_1)^{\sigma_{\alpha,h}} = h_1$, implying that $x = h$. Furthermore, $y_1 = (x_1)^{\sigma_{\alpha,h}} = (x^\alpha)_1$ and $1_1 = (y_1)^{\sigma_{\alpha,h}} = (xy^\alpha)_1$. It follows that $x^\alpha = x^{-1}y$ and $y^\alpha = x^{-1}$.

Recall that

$$H = \mathcal{H}_{p,t,s} = \langle a, b, c \mid a^{p^t} = b^{p^s} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where $t \geq s \geq 1$. We first prove the following claim.

Claim $o(x) = o(y) = o(x^{-1}y) = p^t$ and $\langle x^{p^s} \rangle = \langle y^{p^s} \rangle$.

Since $x^\alpha = x^{-1}y$ and $y^\alpha = x^{-1}$, we have $o(x) = o(y) = o(x^{-1}y)$. Denote by $\exp(H)$ the exponent of H . Since $H = \langle x, y \rangle$, by Lemma 2.1 (3), we have $o(x) = o(y) = o(x^{-1}y) = \exp(H) = p^t$. Note that $H = \langle a, b \rangle = \langle x, y \rangle$. Again by Lemma 2.1 (3), we have $\langle x^{p^s} \rangle \leq \langle a \rangle$ and $\langle y^{p^s} \rangle \leq \langle a \rangle$. Since $o(x) = o(y) = p^t$, we have $\langle x^{p^s} \rangle = \langle y^{p^s} \rangle$, as claimed.

Now we are ready to finish the proof. If $t = s$, then by Lemma 2.1 (4), there exists an automorphism of H sending (x, y) to (a, b) , and by Proposition 3.1 (3), we have $\Gamma \cong \Sigma_{p,t,t}$.

Suppose now that $t > s$. By Claim, we have $\langle x^{p^s} \rangle = \langle y^{p^s} \rangle$. Then there exists $k \in \mathbb{Z}_{p^{t-s}}^*$ such that $y^{p^s} = x^{kp^s}$, and so $(yx^{-k})^{p^s} = 1$. So $o(yx^{-k}) = o(b) = p^s$. By Claim, we have $o(x) = o(a) = p^t$. Since $H = \langle x, y \rangle = \langle x, yx^{-k} \rangle$, by Lemma 2.1 (4), there exists $\gamma \in \text{Aut}(H)$ such that $a^\gamma = x$ and $b^\gamma = yx^{-k}$. It follows that

$$H = \langle x, yx^{-k}, z \mid x^{p^t} = (yx^{-k})^{p^s} = z^p = 1, [x, yx^{-k}] = z, [z, x] = [z, yx^{-k}] = 1 \rangle,$$

and $S = \{1, x, y\} = \{1, x, (yx^{-k})x^k\}$. Clearly, $S^{\gamma^{-1}} = \{1, a, ba^k\}$. By Proposition 3.1 (3), we may assume that $\Gamma = \text{BiCay}(H, \emptyset, \emptyset, \{1, a, ba^k\})$.

Since Γ is normal edge-transitive, by Proposition 4.8, there exists $\sigma_{\theta,g} \in \text{Aut}(\Gamma)_{1_0}$, where $\theta \in \text{Aut}(H)$ and $g \in H$, such that $\sigma_{\theta,g}$ cyclically permutes the three elements in $\Gamma(1_0) = \{1_1, a_1, (ba^k)_1\}$. Without loss of generality, assume that $(\sigma_{\theta,g})_{|\Gamma(1_0)} = (1_1 \ a_1 \ (ba^k)_1)$. Then $a_1 = (1_1)^{\sigma_{\theta,g}} = g_1$, implying that $a = g$. Furthermore, we have

$$(ba^k)_1 = (a_1)^{\sigma_{\theta,g}} = (a^\theta)_1, \quad 1_1 = (ba^k)_1^{\sigma_{\theta,g}} = (a(ba^k)^\theta)_1.$$

Then

$$a^\theta = a^{-1}ba^k = ba^{k-1}c^{-1}, \quad b^\theta = a^{-1}(a^\theta)^{-k} = a^{-1}(ba^{k-1}c^{-1})^{-k}.$$

This implies that $o(a^{-1}(ba^{k-1}c^{-1})^{-k}) = o(b) = p^s$. By Lemma 2.1 (1) and (3), we have $o(a^{k-1}) = p^t$ and $(a^{-1}(ba^{k-1}c^{-1})^{-k})^{p^s} = a^{-(k^2-k+1)p^s} = 1$. It follows that $k^2 - k + 1 \equiv 0 \pmod{p^{t-s}}$, and hence $\Gamma \cong \Sigma_{p,t,s}$. \square

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